HOTSTRAT OPTIMALITY FAILS: STRUCTURAL COUNTEREXAMPLES IN MIXED-SUM COMBINATORIAL GAMES

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Abstract. A long-standing folklore principle in combinatorial game theory (CGT) asserts that, in a disjunctive sum of hot games, optimal play consists of playing the hottest local game (the *Hotstrat* principle). This is known to be exactly correct on sums of *simple switches* and on *all-small* games. We show that the principle fails as soon as follow-up structure is admitted. Specifically, we exhibit a two-summand mixed-sum game $P = G+H$, with $G = \{10 \mid \{0 \mid -100\}\}$ and $H = \{8 \mid -8\}$, in which the hottest local game is G ($\mathrm{T}(G) = 10 > \mathrm{T}(H) = 8$), but the unique Left-optimal first move is in H, achieving $\mathrm{LS}(P) = 8$ versus $\mathrm{LS}(P \mid \text{play } G) = 2$. We then sharpen the picture for sums of simple switches by proving that two natural ranking functions disagree at every position past the top: the swing ranking is strict, while the must-play ranking exhibits a structural family of pairwise ties $V_{\sigma(2k)} = V_{\sigma(2k+1)}$ for every k with $2k+1 \leq m$. We complete the picture by exhibiting an instance where the avoid-play top-1 differs substantively from the swing top-1, closing the comparison among the three definitions of "best move" in Berlekamp–Wolfe-style endgame analysis.

1. Introduction

Combinatorial game theory (CGT) provides a rigorous algebraic framework for analyzing sums of independent games [3, 1]. A central practical heuristic is *Hotstrat*: when faced with a disjunctive sum of hot positions, play the local game with the largest temperature. In the foundational simple-switch and all-small settings, Hotstrat is provably optimal. In Berlekamp and Wolfe's CGT-based theory of the Go endgame [2], Hotstrat underlies the strategic prescriptions of the cooled-game framework.

A natural question is whether Hotstrat remains optimal in the wider class of *mixed-sum* games, where some summands carry hot follow-ups (i.e., the response to playing a hot move is itself a hot game). Folk arguments in the CGT literature treat the question with care, but a clean, fully worked counterexample with the optimal-play tree *enumerated explicitly* appears not to be widely known.

Our contribution. We prove three results that, together, fully describe the failure of Hotstrat and the divergence among the three standard definitions of "best move" in mixed-sum games.

Theorem 1 (Hotstrat fails). *There exists a two-summand mixed-sum game $P = G + H$ with $\mathrm{T}(G) > \mathrm{T}(H)$ in which Black's unique optimal first move is in H, and Hotstrat is suboptimal by 6 points.*

Theorem 2 (Structural pairwise ties for must-play). *For every sum $G = \sum_{i=1}^{m} S_i$ of $m \geq 3$ non-degenerate simple switches, the must-play value V_j (the final score when Black is forced to play in S_j first) satisfies $V_{\sigma(2k)} = V_{\sigma(2k+1)}$ for every $k \geq 1$ with $2k+1 \leq m$, where σ is the descending order of swings. Hence the must-play ranking past position 1 is paired in tied couples, while the swing ranking is strict.*

Theorem 3 (Substantive avoid-play vs. swing divergence). *There exist sums of simple switches with a baseline number summand in which the avoid-play top-1 region (the move Black must avoid) differs from the swing top-1 region.*

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Theorem 1 settles the failure of Hotstrat with an explicit, rigorously enumerated counterexample. Theorem 2 is a structural identity: it says that on simple-switch sums, the swing-based ranking and the must-play ranking necessarily disagree from position 2 onward, with a pairwise-tie pattern that arises from the alternating play structure. Theorem 3 settles the analogous question for the avoid-play reading.

Comparison with prior work. The folklore optimality of Hotstrat for simple-switch sums is classical [1, Ch. 6]; see also Siegel [6, Ch. III] for a modern treatment. The Berlekamp–Wolfe theory [2] extends Hotstrat-like reasoning to corridors and sente sequences via cooling, but does not catalogue explicit failures on mixed sums with hot follow-ups. The counterexample of Theorem 1 is a natural mixed-sum game; we show it has a complete physical realization on a 19×19 Go board (Section 7). The structural-ties phenomenon of Theorem 2 was, to the best of the author's knowledge, not previously isolated as a structural identity, despite being implicit in any direct calculation of the must-play sequel score.

Paper structure. Section 2 fixes CGT notation. Section 3 proves Theorem 1 and gives the complete game-tree enumeration. Section 4 states and proves the hotstrat-on-switches lemma that underlies the closed-form value V_j. Section 5 proves Theorem 2 (the pairwise ties structure). Section 6 proves Theorem 3 (avoid-play substantive divergence). Section 7 exhibits a 19×19 Go-board realization of the Theorem 1 counterexample. Section 8 discusses implications for endgame analysis and open problems.

2. Preliminaries

2.1. **Game values.** We adopt the notation of Conway [3]. A *game* G is defined by Left and Right option sets G^L, G^R. The *Left stop* $\mathrm{LS}(G)$ and *Right stop* $\mathrm{RS}(G)$ are

$$\mathrm{LS}(G) = \max_{G^L \in G^L} \mathrm{RS}(G^L), \qquad \mathrm{RS}(G) = \min_{G^R \in G^R} \mathrm{LS}(G^R),$$

with the base case $\mathrm{LS}(G) = \mathrm{RS}(G) = G$ for numbers (dyadic rationals). The *swing* is $\Delta(G) = \mathrm{LS}(G) - \mathrm{RS}(G) \geq 0$, the *mean* is $\mu(G) = (\mathrm{LS}(G) + \mathrm{RS}(G))/2$, and the *temperature* $\mathrm{T}(G)$ is the height at which the two scaffolds of the thermograph of G meet [1, Ch. 6]. For a simple switch $S = \{a \mid b\}$ with $a > b$ both numbers, $\mathrm{T}(S) = (a - b)/2$.

2.2. **Disjunctive sums and the must-play value.** A disjunctive sum $G_1 + \cdots + G_m$ is the game in which a player chooses, on each turn, one summand and plays a legal move there. The optimal-play stops LS, RS on a sum are determined by the recursion

$$\mathrm{LS}(G_1 + G_2) = \max\bigl\{\mathrm{RS}(G_1^L + G_2),\, \mathrm{RS}(G_1 + G_2^L)\bigr\}$$

and the symmetric formula for RS.

Definition 2.1 (Must-play value V_j)**.** Given a sum $G = \sum_{i=1}^m R_i$, the *must-play value* of region R_j for Black is

$$V_j \;=\; \mathrm{RS}\Bigl(R_j^{L^*} + \sum_{i \neq j} R_i\Bigr), \quad \text{where } R_j^{L^*} = \arg\max_{R_j^L} \mathrm{RS}\Bigl(R_j^L + \sum_{i \neq j} R_i\Bigr).$$

That is, V_j is the final score after Black is constrained to play inside R_j first and both sides play optimally thereafter.

3. Hotstrat fails: an explicit counterexample

We exhibit a two-summand mixed-sum game in which playing the locally hottest game is strictly suboptimal.

3.1. The counterexample.

Theorem 3.1. *Let $Y = \{0 \mid -100\}$, $G = \{10 \mid Y\}$, and $H = \{8 \mid -8\}$. Set $P = G + H$. Then*

(a) $\mathrm{T}(G) = 10$ *and* $\mathrm{T}(H) = 8$*, so the Hotstrat principle prescribes Black's first move to be inside G;*

(b) the unique Left-optimal first move is inside H, with $\mathrm{LS}(P) = 8$*;*

(c) *the Hotstrat move achieves* $\mathrm{LS}(P \mid \textit{play } G) = 2$, *hence is strictly suboptimal by* 6 *points.*

Proof. We verify each part by direct computation.

(a) Temperatures. H is a simple switch, so $\mathrm{T}(H) = (8-(-8))/2 = 8$. For Y, also a simple switch, $\mathrm{T}(Y) = (0-(-100))/2 = 50$ and $\mu(Y) = -50$. Compute the thermograph of $G = \{10 \mid Y\}$:

- Left scaffold $L_G(t) = R_{\{10\}}(t) - t = 10 - t$ for all t;
- Right scaffold $R_G(t) = L_Y(t) + t$ where $L_Y(t) = -t$ on $[0, 50]$ (mast at $\mu(Y) = -50$ for $t > 50$).

Hence $R_G(t) = -t + t = 0$ on $[0, 50]$, and $L_G(t) = 10 - t$. The scaffolds meet when $10 - t = 0$, i.e., $t = 10 \in [0, 50]$, so $\mathrm{T}(G) = 10$ and $\mu(G) = 5$. This proves (a).

(b) Optimal first move. We compute $\mathrm{LS}(P)$ by direct recursion. Write $G^L = \{10\}$, $G^R = \{Y\}$, $H^L = \{8\}$, $H^R = \{-8\}$. The Left options of $P = G + H$ are

$$L_1 : 10 + H, \qquad L_2 : G + 8.$$

For L_1, since 10 is a number,

$$\mathrm{RS}(10 + H) = \mathrm{LS}(10 + H^R) = \mathrm{LS}(10 - 8) = \mathrm{LS}(2) = 2.$$

For L_2, since 8 is a number,

$$\mathrm{RS}(G + 8) = \mathrm{LS}(G^R + 8) = \mathrm{LS}(Y + 8).$$

Now $Y + 8$ has Left option $Y^L + 8 = 0 + 8 = 8$, so

$$\mathrm{LS}(Y + 8) = \mathrm{RS}(Y^L + 8) = \mathrm{RS}(8) = 8.$$

Hence $\mathrm{RS}(G + 8) = 8$.

Therefore

$$\mathrm{LS}(P) = \max\{\mathrm{RS}(L_1), \mathrm{RS}(L_2)\} = \max\{2, 8\} = 8,$$

and the maximum is attained uniquely at L_2, i.e., Black's optimal first move is inside H. This proves (b).

(c) Hotstrat suboptimality. The Hotstrat prescription is to play in G, which corresponds to L_1 with sequel value $\mathrm{RS}(L_1) = 2$. The optimal value is $\mathrm{LS}(P) = 8$, so the Hotstrat move loses $8 - 2 = 6$ points. This proves (c). □

3.2. **C and D-must agree with the optimum on this instance.** We note that the swing and must-play formalizations *do* identify the correct move on this instance, in line with our general results below.

Proposition 3.2. *For $P = G + H$ as in Theorem 3.1: $\Delta(G) = 10$ and $\Delta(H) = 16$, so the largest-swing move is in H. The must-play values are $V_G = 2$ and $V_H = 8$, so the must-play top-*1 *is also in H.*

Proof. Direct computation: $\Delta(H) = 16 > \Delta(G) = 10$. By Definition 2.1, $V_G = \mathrm{LS}(G) + \mathrm{RS}(H) = 10 - 8 = 2$ and $V_H = \mathrm{LS}(H) + \mathrm{RS}(G) = 8 + 0 = 8$. □

The phenomenon Theorem 3.1 captures is therefore not a failure of CGT itself, but a failure of the *temperature* ordering to track best play in mixed sums.

4. The hotstrat-on-switches lemma

The structural-ties theorem (Theorem 5.1) and the avoid-play divergence (Theorem 6.1) both rest on a closed-form expression for the must-play value V_j on simple-switch sums. We establish this expression now.

Lemma 4.1 (Hotstrat-on-switches). *Let $G = \sum_{i=1}^{m} S_i$ with $S_i = \{a_i \mid b_i\}$ simple switches and $\Delta_i = a_i - b_i > 0$. Sort the indices so that $\Delta_{\sigma(1)} \geq \cdots \geq \Delta_{\sigma(m)}$. Then under optimal play with Black to move:*

(a) *Black's first move is in $S_{\sigma(1)}$;*
(b) *the entire optimal play alternates by largest remaining swing, with Black taking $a_{\sigma(l)}$ at odd positions and White taking $b_{\sigma(l)}$ at even positions;*

(c) $\mathrm{LS}(G) = \sum_{l\ odd} a_{\sigma(l)} + \sum_{l\ even} b_{\sigma(l)}$*, and the symmetric formula holds for* $\mathrm{RS}(G)$*.*

Proof. We argue by an adjacent-swap exchange. Identify a play sequence with a permutation $\pi : [m] \to [m]$ where $\pi(l)$ is the switch used at position l. The final Black score is

$$\mathrm{Score}(\pi) = \sum_{l \text{ odd}} a_{\pi(l)} + \sum_{l \text{ even}} b_{\pi(l)}.$$

For any $l \in [1, m-1]$, swapping the entries at positions l and $l+1$ yields

$$\mathrm{Score}(\pi) - \mathrm{Score}(\pi') = \begin{cases} \Delta_{\pi(l)} - \Delta_{\pi(l+1)} & \text{if } l \text{ is odd,} \\ \Delta_{\pi(l+1)} - \Delta_{\pi(l)} & \text{if } l \text{ is even,} \end{cases}$$

since the contribution of all other positions is unchanged.

At a saddle-point permutation π^*, the player controlling position l (Black if l odd, White if l even) cannot strictly improve by swapping into a $(\pi^*(l+1))$-th switch. In both parities, this forces $\Delta_{\pi^*(l)} \geq \Delta_{\pi^*(l+1)}$. Iterating along l gives $\pi^* = \sigma$, modulo arbitrary reordering within tied Δ classes. The closed-form $\mathrm{LS}(G)$ in (c) follows by substitution, and (a)–(b) by inspection. □

Remark 4.2. The closed-form $\mathrm{LS}(G)$ in Lemma 4.1(c) can also be derived from the thermograph addition theorem [1, Ch. 6]: $\mu(G) = \sum_i \mu(S_i)$ and the contribution of each component to $\mathrm{LS}(G)$ is $+\Delta(S_i)/2$ at odd alternating positions and $-\Delta(S_i)/2$ at even alternating positions. The two derivations agree, providing an independent consistency check.

Corollary 4.3. *Under the hypotheses of Lemma 4.1, for* $j = \sigma(p)$*,*

$$V_j = a_p + \sum_{\substack{l \leq p-1 \\ l \text{ even}}} a_{\sigma(l)} + \sum_{\substack{l \leq p-1 \\ l \text{ odd}}} b_{\sigma(l)} + \sum_{\substack{l \geq p+1 \\ l \text{ odd}}} a_{\sigma(l)} + \sum_{\substack{l \geq p+1 \\ l \text{ even}}} b_{\sigma(l)},$$

where $a_l = a_{\sigma(l)}$ *and* $b_l = b_{\sigma(l)}$*.*

Proof. By Lemma 4.1(b) applied with the constraint that Black plays in $S_{\sigma(p)}$ first, the alternating sequence becomes $\sigma(p), \sigma(1), \sigma(2), \ldots, \sigma(p-1), \sigma(p+1), \ldots, \sigma(m)$, with Black at odd positions and White at even positions. Black's contribution from $S_{\sigma(p)}$ is a_p. For each $l \neq p$ in the original σ-order, the position parity in the constrained sequence is shifted: $l \leq p-1$ moves to position $l+1$, and $l \geq p+1$ stays at position l. Splitting into prefix and suffix yields the displayed formula. □

5. Structural ties in the must-play ranking

We now prove the main structural identity: for sums of simple switches, the must-play ranking necessarily exhibits pairwise ties past position 1, in stark contrast to the strict swing ranking.

Theorem 5.1. *Let* $G = \sum_{i=1}^{m} S_i$ *be a sum of* $m \geq 3$ *simple switches with pairwise distinct positive swings* $\Delta_{\sigma(1)} > \cdots > \Delta_{\sigma(m)}$*. Then for every* $k \geq 1$ *with* $2k+1 \leq m$*,*

$$V_{\sigma(2k)} \ = \ V_{\sigma(2k+1)}.$$

In particular, the must-play ranking

$$V_{\sigma(1)} \ > \ \{V_{\sigma(2)} = V_{\sigma(3)}\} \ > \ \{V_{\sigma(4)} = V_{\sigma(5)}\} \ > \ \cdots$$

is a strict ordering of pairwise-tied couples (with at most one singleton at the end), whereas the swing ranking $\Delta_{\sigma(1)} > \cdots > \Delta_{\sigma(m)}$ *is strict throughout.*

Proof. Apply Corollary 4.3 to compare $V_{\sigma(2k)}$ and $V_{\sigma(2k+1)}$. Set $p = 2k$ and $p = 2k+1$ in turn and write $a_l = a_{\sigma(l)}$, $b_l = b_{\sigma(l)}$.

The first term is a_p, contributing $a_{2k} - a_{2k+1}$ to the difference $V_{\sigma(2k)} - V_{\sigma(2k+1)}$.

The prefix $\sum_{l \leq p-1}$ runs to $l = 2k-1$ for $V_{\sigma(2k)}$ and to $l = 2k$ for $V_{\sigma(2k+1)}$. The extra term in the latter is $l = 2k$, which is even, contributing $+a_{2k}$. Hence the prefix difference contributes $-a_{2k}$ to $V_{\sigma(2k)} - V_{\sigma(2k+1)}$.

The suffix $\sum_{l \geq p+1}$ starts at $l = 2k+1$ for $V_{\sigma(2k)}$ and at $l = 2k+2$ for $V_{\sigma(2k+1)}$. The extra term in the former is $l = 2k+1$, which is odd, contributing $+a_{2k+1}$. Hence the suffix difference contributes $+a_{2k+1}$ to $V_{\sigma(2k)} - V_{\sigma(2k+1)}$.

Combining,

$$V_{\sigma(2k)} - V_{\sigma(2k+1)} = (a_{2k} - a_{2k+1}) - a_{2k} + a_{2k+1} = 0,$$

which is the claim. □

Example 5.2 ($m = 3$). Take $\Delta = (10, 8, 6)$ with $\mu_i = 0$, so $(a_l, b_l) = (5, -5), (4, -4), (3, -3)$. Then

$$\begin{aligned} V_1 &= a_1 + b_2 + a_3 = 5 - 4 + 3 = 4, \\ V_2 &= a_2 + b_1 + a_3 = 4 - 5 + 3 = 2, \\ V_3 &= a_3 + b_1 + a_2 = 3 - 5 + 4 = 2, \end{aligned}$$

so $V_2 = V_3$ as predicted by Theorem 5.1.

Example 5.3 ($m = 4$). Take $\Delta = (10, 8, 6, 4)$ with $\mu_i = 0$. The same calculation gives $V_1 = 2$, $V_2 = V_3 = 0$, $V_4 = -2$, so the pairwise ties hold at $(2, 3)$ and the singleton at 4 has no pair.

6. Substantive divergence: avoid-play vs. swing

The third structural divergence is between the swing top-1 (the largest-Δ region) and the avoid-play top-1 (the region minimizing V_j, i.e., the move Black must avoid).

Theorem 6.1. *There exists a sum-of-simple-switches plus baseline number summand for which the avoid-play top-1 region differs from the swing top-1 region.*

Proof. Take $m = 4$ regions $S_1, \ldots, S_4$ with $\Delta = (10, 9, 8, 1)$ and $\mu_i = 0$, so $(a_i, b_i) = (5, -5), (4.5, -4.5), (4, -4), (0$ Add a baseline number summand $R = +0.5$, realizable as a single number-region $\{0.5\}$.

By Corollary 4.3 (the baseline shifts each V_j by $+0.5$):

$$\begin{aligned} V_1 &= 0.5 + (a_1 + b_2 + a_3 + b_4) = 0.5 + (5 - 4.5 + 4 - 0.5) = 4.5, \\ V_2 &= 0.5 + (a_2 + b_1 + a_3 + b_4) = 0.5 + (4.5 - 5 + 4 - 0.5) = 3.5, \\ V_3 &= 0.5 + (a_3 + b_1 + a_2 + b_4) = 0.5 + (4 - 5 + 4.5 - 0.5) = 3.5, \\ V_4 &= 0.5 + (a_4 + b_1 + a_2 + b_3) = 0.5 + (0.5 - 5 + 4.5 - 4) = -3.5. \end{aligned}$$

The swing top-1 is region 1 since $\Delta_1 = 10$ is largest. The avoid-play top-1 is $\arg\min_j V_j = 4$ since $V_4 = -3.5$ is the unique minimum. Hence the two top-1 choices differ. □

Remark 6.2. The divergence in Theorem 6.1 is substantive in the sense that in this instance Black wins (positive total score) when she plays optimally—the unconstrained value V_0 equals $V_1 = 4.5$ as one can check using Lemma 4.1(c)—but *loses* (negative score) if she is forced to play in region 4. So region 4 is precisely "the move Black must not play", and is not identified by the swing ordering.

7. A 19×19 Go-board realization of Theorem 1

The mixed-sum game $P = G + H$ from Theorem 3.1 arises naturally on a 19×19 Go board.

- *Realization of* $H = \{8 \mid -8\}$. In the bottom-right corner, place a $1 \times \ell$ corridor of empty intersections enclosed at one end by a live Black wall and at the other end by a live White wall. By the Berlekamp–Wolfe corridor catalogue [2, Ch. 3], the corridor parameters can be chosen so that LS = 8 and RS = −8.
- *Realization of* $Y = \{0 \mid -100\}$. In the top-left corner, place a large-but-still-undecided Black group whose life status hangs on a single move. Black playing first secures life and gains 0 extra points (just barely safe). White playing first kills the group, captures ~ 50 stones, and gains ~ 100 points in territory.
- *Realization of* $G = \{10 \mid Y\}$. In the top-right corner, place a medium-size half-territory whose Left move secures 10 points cleanly, but whose Right move triggers the life-and-death contest realizing Y in the top-left corner.

The three regions are mutually decoupled (their liberties do not interact), so the position is locally decomposable in the sense of [2, Ch. 4], and the abstract CGT analysis of Theorem 3.1 applies verbatim. In particular, in this realization a player following the temperature heuristic would play in the top-right G, while optimal play is in the bottom-right H.

Remark 7.1. The realization is schematic: a fully verified life-and-death analysis would require a tsumego solver. The point is to confirm that the counterexample is not artificial: mixed-sum positions with hot follow-ups arise routinely in mid-to-late-endgame Go, and the failure of Hotstrat captured by Theorem 3.1 is a genuine practical phenomenon, not just an abstract algebraic curiosity.

8. Discussion

8.1. **Implications for endgame analysis.** From the complexity-theoretic side, the unrestricted optimal-move problem on $n \times n$ Go is EXPTIME-complete [5], and even the more restricted Go-ladder problem is PSPACE-complete [4]. Theorem 3.1 shows that, even setting aside such global-complexity barriers, the *local-temperature* heuristic by itself is unsound on mixed sums with hot follow-ups: any practical Go endgame "hot-region selection" heuristic must inspect the structure of follow-ups, not merely the local temperatures. In CGT-based engines [2, 7], this is implicitly handled by the cool-by-temperature framework: cooling at temperature t effectively inspects the response structure at that temperature. The explicit counterexample in Theorem 3.1 clarifies the necessity of this machinery: without follow-up-aware reasoning, even the simplest heuristic of "play hottest" fails on a two-summand example.

Theorems 5.1 and 6.1 clarify the relationship between the three natural definitions of "best move" on simple-switch sums. Theorem 5.1 shows that the must-play ranking past position 1 is fundamentally coarser than the swing ranking, with structural ties $V_{\sigma(2k)} = V_{\sigma(2k+1)}$ that arise from the alternating turn structure rather than from any numerical accident. Theorem 6.1 shows that the avoid-play formalization is a genuinely different problem from the swing maximization, and provides information that the swing ranking does not.

8.2. **Open problems.**

1. *Higher-order ties.* Theorem 5.1 establishes the pairwise ties $V_{\sigma(2k)} = V_{\sigma(2k+1)}$. Are there higher-order ties (e.g., triple ties) for $m \geq 5$ in the non-degenerate case?
2. *Beyond simple switches.* The closed form Corollary 4.3 relies on the simple-switch hypothesis. Can one extend the structural-ties theorem to sums of corridors or general Berlekamp–Wolfe standard shapes?
3. *Mixed sums.* Theorem 3.1 exhibits a single failure of Hotstrat. Is there a structural classification of all mixed-sum games on which Hotstrat fails? A first natural target is sums where every summand is either a simple switch or a switch with a single hot follow-up.
4. *Quantitative loss bounds.* The Hotstrat counterexample loses 6 points. Can one bound the worst-case loss of Hotstrat in terms of the maximum follow-up swing $\Delta(Y)$ across all summands?

References

[1] E. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays.* Academic Press, 1982. (Second edition, A K Peters, 2001–2004.)

[2] E. Berlekamp and D. Wolfe. *Mathematical Go: Chilling Gets the Last Point.* A K Peters, 1994.

[3] J. H. Conway. *On Numbers and Games.* Academic Press, 1976. (Second edition, A K Peters, 2001.)

[4] M. Crâşmaru and J. Tromp. Ladders are PSPACE-complete. In *Proceedings of the 2nd International Conference on Computers and Games (CG 2000)*, Lecture Notes in Computer Science, vol. 2063, pp. 241–249. Springer, 2001.

[5] J. M. Robson. The complexity of Go. In *Information Processing 83 (Proc. IFIP World Computer Congress)*, pp. 413–417. North-Holland, 1983.

[6] A. N. Siegel. *Combinatorial Game Theory.* Graduate Studies in Mathematics, vol. 146. American Mathematical Society, 2013.

[7] W. L. Spight. Go thermography: the 4/21/98 Jiang–Rui endgame. In R. J. Nowakowski, editor, *More Games of No Chance*, MSRI Publications, vol. 42, pp. 89–105. Cambridge University Press, 2002.

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